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GAUGE FIELDS AS RINGS OF GLUE

A.M. POLYAKOV
Landau Institute for Theoretical Physics, USSR

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In this paper we take the view that gauge fields can be considered as chiral fields on a loop space, both in classical and in quantum theories. As a result, gauge interactions are interpreted as propagation of the infinitely thin rings formed by the lines of color-electric flux. Equations of motion governing this propagation are derived. In the three-dimensional case some higher conserved currents in the loop space are obtained, indicating that hidden symmetry is present in the theory. In the four-dimensional case the question of hidden symmetry remains unclear. Ward identities in the loop space are obtained and their mathematical structure is investigated. Possible extensions and applications of these results are discussed.

1. Introduction

There can be no doubt now that gauge fields play a very fundamental role in Nature. It is even conceivable that all existing interactions are mediated by gauge bosons. At the same time the theory of gauge fields is not in good shape. Of course we know how to perform some standard things like a perturbation theory, and we even have some understanding of new concepts, like topological excitations of fields. But we still lack methods for answering questions like which subgroups of a gauge group are confining, and which subgroups are spontaneously broken. These questions are of primary importance both for quantum chromodynamics and quantum flavordynamics. So, a better understanding of gauge-field dynamics seems essential to future progress.

In this paper I shall try to present some small steps in this direction. Though the results, if any, are very modest, I believe that the methods and concepts with which I work are basically adequate and may lead eventually to some valuable understanding.

The basic idea is that gauge fields can be considered as chiral fields, defined on the space of all possible contours (the loop space) [1]. The origin of the idea lies in the expectation that, in the confining phase of a gauge theory, *closed* strings should play the role of *elementary* excitations [2,3]. In contrast, in conventional field theories, the elementary excitations are just point-like particles. This observation makes the formulation of the basic equations in loop space both natural and adequate. As a result we get equations that reveal a striking similarity between chiral and gauge fields. The analogy exists both in quantum and in classical theories. In the case of chiral fields the interaction can be understood in terms of propagation and collisions of massless Goldstone particles, which acquire mass (and thus restore the symmetry). We will show that in a similar fashion the interaction of gauge fields can be represented as a propagation of infinitely thin bare strings with zero slope that eventually acquire non-zero slope.

One can hope that since these gauge strings arise from a theory that is in many respects unique, the interaction between strings must be very special and possess some hidden symmetry. This hope is substantiated by the fact that ordinary chiral fields do have such a symmetry, being in fact completely integrable (that means that the number of conserved integrals is equal to the number of degrees of freedom).

Whether this is really the case with gauge fields is not completely clear. It has been shown [1] on the classical level in three-dimensional Yang-Mills theory that there exists a set of conserved currents in the loop space. At the same time, after many attempts I am convinced that there are no non-trivial, conserved (in the usual sense) integrals in gauge theories. However, it is the functionally conserved currents that are relevant for string interactions. The main unresolved problem is the extension of these results to four dimensions and to quantum theory. Not being able to resolve this problem, we describe in this paper some possible approaches to it.

Another possible use of our representation of gauge theories might be its applications to different approximation schemes (such as large-N or strong coupling expansions). We will comment on that later.

Since two-dimensional chiral fields play an important role in our discussion we begin this paper with a short review of their properties (sect. 2). Then we discuss classical conserved currents in gauge theories (sect. 3). After that (sect. 4) an analysis of the renormalization properties of fields on loop space is given. In sect. 5 we derive the quantum equations of motion and Ward identities in loop space. In sect. 6 we analyze some new mathematical objects that appear in loop space, such as the δ-function, integration, etc. At the end of this paper we speculate about the directions of future progress.

2. Chiral fields in two dimensions

Chiral fields are described by the matrix field $g(x) \in G$, where G is some Lie group. The lagrangian is given by

$$L = \frac{1}{2e_0^2} \mathrm{Tr}(\partial_\mu g^{-1}(x)\, \partial_\mu g(x))$$

$$= \frac{1}{2e_0^2} \operatorname{Tr} A_\mu^2(x) , \tag{2.1}$$

$$A_\mu(x) = g^{-1}(x)\, \partial_\mu g(x) .$$

The equations of motion for the lagrangian (2.1) are given by

$$\partial_\mu A_\mu(x) = 0 ,$$
$$\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu, A_\nu] = 0 . \tag{2.2}$$

The second equation in (2.2) expresses the geometrical constraint (zero curvature) that follows from the definition of A_μ. In the case of two-dimensional space-time this theory is known to be completely integrable [4,5]. That means that there are infinitely many non-trivial conserved currents in it. For the sake of completeness and for establishing notation I will demonstrate this fact briefly. The first current, $J_\mu^{(1)}(x)$, is just $A_\mu(x)$ itself. To find the second one $J_\mu^{(2)}(x)$, let us study the expression

$$J_\mu^{(2)}(x) = \epsilon_{\mu\nu} A_\nu \chi(x) + [A_\mu(x), \chi(x)] , \tag{2.3}$$

where $\chi(x)$ is to be determined. For conservation of $J_\mu^{(2)}(x)$ we require

$$\partial_\mu J_\mu^{(2)}(x) = \epsilon_{\mu\nu} \partial_\mu A_\nu + [A_\mu, \partial_\mu \chi]$$
$$= [A_\mu, \partial_\mu \chi - \tfrac{1}{2} \epsilon_{\mu\nu} A_\nu] = 0 . \tag{2.4}$$

So, we have to define $\chi(x)$ by

$$\chi(x) = \tfrac{1}{2} \int^x \epsilon_{\mu\nu} A_\nu \, \mathrm{d}x_\mu , \tag{2.5}$$

where by the first of eqs. (2.2) the function χ is independent of the contour of integration in (2.4). In order to get a general expression for the higher currents, one notices that eqs. (2.2) are the consequence of the consistency of the Lax pair of equations [4,5]:

$$(\partial_\mu + \gamma \epsilon_{\mu\nu} \partial_\nu)\, \phi = [A_\mu, \phi] \tag{2.6}$$

Here γ is arbitrary parameter. It is trivial to check that the compatibility condition (equality of cross second derivatives)

$$[\partial_\mu + \gamma \epsilon_{\mu\lambda} \partial_\lambda - A_\mu, \partial_\nu + \gamma \epsilon_{\nu\lambda} \partial_\lambda - A_\nu] = 0 \tag{2.7}$$

is equivalent to the system (2.2). To find the currents, we use the expansion

$$\phi(x, \gamma) = 1 + \sum_{k=1}^{\infty} \phi_k(x)\, \gamma^{-k} . \tag{2.8}$$

From (2.6) we have

$$\epsilon_{\mu\nu} \partial_\nu \phi_k = -[(\partial_\mu - A_\mu) , \phi_{k-1}] . \tag{2.9}$$

Then, the conserved currents are given by

$$J_\mu^{(k)}(x) = \epsilon_{\mu\nu}\partial_\nu\phi_k \,. \tag{2.10}$$

It is known that the existence of these currents determines the S-matrix of the chiral fields [6]. Now let us comment on the quantum version of this theory. Two things are known. First, analyses of the anomalies in the continuum limit [7,8] have shown that the currents remain conserved, although the currents themselves are modified. These analyses neglected possible topological anomalies, i.e., all total divergences. Second, I have found the Lax representation for the lattice version of the chiral fields. However, this representation exists only for the general linear group GL(n, R). After several attempts I am convinced that the chiral fields on the lattice with O(n) or SU(n) groups are not integrable. All this implies that one should be very careful with topological anomalies in the continuum limit. They present a problem still to be resolved.

3. Loop space

Now let us proceed to our main object, gauge fields. In establishing the above-mentioned analogy, the basic role is played by a well-known object, the element of the holonomy group

$$\psi(\mathrm{c}) = \mathrm{P}\exp \oint_{\mathrm{c}} A_\mu \,\mathrm{d}x^\mu \,. \tag{3.1}$$

Here c is some closed contour that begins at some fixed point x_0 and P stands for Dyson ordering along this contour. $\psi(\mathrm{c})$ depends on x_0, but this will be implicit in our notation. Let us consider now $\psi(\mathrm{c})$ as a chiral field, and introduce the connection in the loop space by the formula

$$F_\mu(s, \mathrm{c}) = \frac{\delta\psi(\mathrm{c})}{\delta x_\mu(s)}\psi^{-1}(\mathrm{c}) \,. \tag{3.2}$$

Here the contour c is parametrized by the function $x_\mu(s)$, $\psi(\mathrm{c}) = \psi(x(s))$. This functional should not depend on the way we parametrize $x(s)$, and hence

$$\psi[x(\phi(\tilde{s}))] = \psi[x(s)] \,, \tag{3.3}$$

$$\frac{\mathrm{d}x_\mu}{\mathrm{d}s}\frac{\delta\psi}{\delta x_\mu(s)} = 0 \,, \qquad \frac{\mathrm{d}x_\mu}{\mathrm{d}s}F_\mu(s,\mathrm{c}) = 0 \,.$$

From the definition (3.2) we conclude that

$$\frac{\delta F_\mu(s,\mathrm{c})}{\delta x_\nu(s^1)} - \frac{\delta F_\nu(s^1,\mathrm{c})}{\delta x_\mu(s)} + [F_\mu(s,\mathrm{c}), F_\nu(s^1,\mathrm{c})] = 0 \,. \tag{3.4}$$

The important result, which we will demonstrate now, is that the Yang-Mills equations take a very simple form in terms of F_μ. To show this, we notice that

$$\delta\psi(c) = \int_0^{2\pi} ds\, P[\exp \int_0^s A_\mu\, dx_\mu]\, F_{\mu\nu}(x(s))\, \frac{dx_\nu(s)}{ds}\, P[\exp \int_s^{2\pi} A_\mu\, dx_\mu]\, \delta x_\mu(s)\,. \tag{3.5}$$

From this we derive

$$\frac{\delta\psi}{\delta x_\mu(s)}\,\psi^{-1} = \Omega(s, c)\, F_{\mu\nu}(x(s))\, \Omega^{-1}(s, c)\, \frac{dx_\nu(s)}{ds}\,. \tag{3.6}$$

Here

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,,$$

$$\Omega(s, c) = P \exp \int_0^s A_\mu\, dx_\mu\,.$$

We see that $F_\mu(s, c)$ has a natural interpretation: it is a Lorentz force, transported parallel to the initial point of our contour. From (3.6) it is easy to get the following identity:

$$\frac{\delta F_\mu(s, c)}{\delta x_\mu(s^1)} + [F_\mu(s^1), F_\mu(s)] = \Omega(s, c)(\nabla_\mu F_{\mu\nu}(x(s)))\, \Omega^{-1}(s, c)\, \frac{dx_\nu(s)}{ds} \times \delta(s - s^1)\,, \tag{3.7}$$

for $s^1 \leqslant s$. Comparing with (3.5) and using the Yang-Mills equations, we find the following results for $F_\mu(s, c)$:

$$\frac{\delta F_\mu(s, c)}{\delta x_\nu(s^1)} - \frac{\delta F_\nu(s^1, c)}{\delta x_\mu(s)} + [F_\mu(s, c), F_\nu(s^1, c)] = 0\,,$$

$$\frac{dx_\mu(s)}{ds}\, F_\mu(s, c) = 0\,,$$

$$\frac{\delta F_\mu(s, c)}{\delta x_\mu(s)} = 0 \tag{3.8}$$

The interpretation of eqs. (3.8) becomes obvious after comparing them with eqs. (2.2). We see that the gauge fields with non-zero field strength (or curvature) in ordinary space define a chiral field with zero curvature in loop space. It is very natural to expect, then, an infinite number of contour currents, analogous to (2.9), satisfying the equations

$$\frac{\delta J_\mu^{(k)}(s, c)}{\delta x_\mu(s)} = 0\,. \tag{3.9}$$

To verify this expectation, we first consider three-dimensional space-time. Just as before we identify $J_\mu^{(1)}(s, \mathrm{c})$ with $F_\mu(s, \mathrm{c})$ and look for $J_\mu^{(2)}(s, \mathrm{c})$ in the form

$$J_\mu^{(2)}(s, \mathrm{c}) = \epsilon_{\mu\nu\lambda} \frac{\mathrm{d}x_\nu(s)}{\mathrm{d}s} F_\lambda(s, \mathrm{c}) + [F_\mu(s, \mathrm{c}), \chi(s, \mathrm{c})] \; . \tag{3.10}$$

From this we find

$$\frac{\delta J_\mu^{(2)}(s, \mathrm{c})}{\delta x_\mu(s)} = \epsilon_{\mu\nu\lambda} \frac{\mathrm{d}x_\nu(s)}{\mathrm{d}s} \frac{\delta F_\lambda(s, \mathrm{c})}{\delta x_\mu(s)} + \left[F_\mu(s, \mathrm{c}), \frac{\delta\chi}{\delta x_\mu(s)}\right]$$

$$= -\epsilon_{\mu\nu\lambda} \frac{\mathrm{d}x_\nu}{\mathrm{d}s} [F_\mu(s), F_\lambda(s)] + \left[F_\mu(s, \mathrm{c}), \frac{\delta\chi}{\delta x_\mu}\right] = 0 \; . \tag{3.11a}$$

For eq. (3.11a) to be satisfied it is sufficient to find a $\chi(s, \mathrm{c})$ such that

$$\frac{\delta\chi(s, \mathrm{c})}{\delta x_\mu(s)} = \epsilon_{\mu\nu\lambda} \frac{\mathrm{d}x_\nu(s)}{\mathrm{d}s} F_\lambda(s, \mathrm{c}) + g(s, \mathrm{c}) \frac{\mathrm{d}x_\mu}{\mathrm{d}s}. \tag{3.11b}$$

(Here g is an arbitrary functional.) Now we have to prove that eq. (3.11b) is indeed consistent, i.e., that the functional $\chi(s, \mathrm{c})$ exists. It is necessary to check that the quantities $\delta^2\chi(s, \mathrm{c})/\delta x_\mu(s')\, \delta x_\nu(s'')$ possess $\mu \to \nu$, $s' \to s''$ symmetry, or, better, that this symmetry is consistent with eq. (3.11b). Let us notice that from (3.11b) it follows that

$$F_\mu(s, \mathrm{c}) = \epsilon_{\mu\nu\lambda} t_\nu(s) \frac{\delta\chi(s, \mathrm{c})}{\delta x_\lambda(s)} \,, \qquad t_\nu(s) = \frac{\mathrm{d}x_\nu}{\mathrm{d}s}. \tag{3.12}$$

Taking the variational divergence and noticing that differentiation of $t_\nu(s)$ gives no contribution, we get

$$\frac{\delta F_\mu(s, \mathrm{c})}{\delta x_\mu(s)} = 0 \; . \tag{3.13}$$

So we see that, due to the equations of motion, the necessary condition for consistency is satisfied. The sufficiency of this condition can also be proved. In principle it is not difficult to derive the higher conserved currents. Just as in case of two-dimensional models they are contained in the identity

$$\left(\frac{\delta}{\delta z_\mu(s)} + \gamma\epsilon_{\mu\nu\lambda} t_\nu(s) \frac{\delta}{\delta z_\lambda(s)}\right) \phi(s, \mathrm{c}, \gamma) = [F_\mu(s, \mathrm{c}), \phi(s, \mathrm{c})] \; , \tag{3.14}$$

where γ is an arbitrary parameter. Again the necessary and sufficient condition for the function $\phi(s, \mathrm{c}, \gamma)$ to exist is the fulfillment of the equations of motion. In the derivation of the conservation laws, one has to remember that second derivatives of the functionals are usually singular (*cf.* eq. (6.1)) but it is possible to check that these singularities do not effect our derivation.

4. Renormalization of the loop fields

In this section we shall consider the problem of divergences and renormalization for the contour averages. It is far from trivial since the loop fields are non-local. So the best thing we can do is exploit ordinary perturbation theory and analyze all possible divergences in this framework. Let us begin with the first contribution to $\langle \mathrm{Tr}\ \psi(\mathrm{c})\rangle$, gives by the diagram of fig. 1. It is given by

$$\langle \mathrm{Tr}\ \psi(\mathrm{c})\rangle^{(1)} \sim \oint_{\mathrm{c}}\oint_{\mathrm{c}} \frac{\mathrm{d}x_\mu\, \mathrm{d}x'_\mu}{(x-x')^2} \equiv f(\mathrm{c})\,. \tag{4.1}$$

The integral here is clearly divergent. If the theory were cut off, we would have

$$f_{\mathrm{reg}}(\mathrm{c}) = \oint_{\mathrm{c}}\oint_{\mathrm{c}} \frac{\mathrm{d}x_\mu\, \mathrm{d}x'_\mu}{(x-x')^2 + a^2} = \int \frac{\dot{x}(s)\,\dot{x}(s+t)\, \mathrm{d}s\, \mathrm{d}t}{[x(s+t) - x(s)]^2 + a^2}\,. \tag{4.2}$$

(Here a^{-1} is the cutoff.) It is very convenient to chose a parametrization such that

$$\dot{x}^2 \equiv \left(\frac{\mathrm{d}x_\mu}{\mathrm{d}s}\right)^2 = \mathrm{const}\,, \qquad \dot{x}\cdot\ddot{x} = 0\,. \tag{4.3}$$

Expanding the integrand in (4.2) in powers of t we get

$$\begin{aligned} f_{\mathrm{reg}}(\mathrm{c}) &= \int \mathrm{d}s\,\dot{x}^2(s) \int \frac{\mathrm{d}t}{\dot{x}^2(s)\, t^2 + a^2} + \text{finite terms} \\ &= \frac{\pi}{a}\int \mathrm{d}s\sqrt{\dot{x}^2} + \text{finite terms} = \frac{\pi}{a} L(\mathrm{c}) + \text{finite terms}\,. \end{aligned} \tag{4.4}$$

(Here $L(\mathrm{c})$ is the length of c.) The physical meaning of (4.4) is very simple. Recall that $\langle \mathrm{Tr}\ \psi(\mathrm{c})\rangle$ can be understood as the effective action for a test particle being guided along the trajectory c. The divergent part of this effective action is just the mass renormalization of the test particle. So, this argument leads us to the conjecture that

$$\langle \mathrm{Tr}\ \psi(\mathrm{c})\rangle = \mathrm{e}^{-\gamma L(\mathrm{c})/a} G_{\mathrm{ren}}(\mathrm{c})\,, \tag{4.5}$$

where $G_{\mathrm{ren}}(\mathrm{c})$ is finite, provided it is expressed in terms of the renormalized coupling constant. We shall analyze this conjecture a little later. Now we have to remark that eq. (4.4) is true only for the smooth contours. If the function $\mathrm{d}x_\mu/\mathrm{d}s$ has jumps, the result will be different. Let us consider for example the contour in fig. 2 with

$\langle \mathrm{Tr}\ \Psi(\mathrm{C})\rangle^{(1)} =$ 

Fig. 1. First perturbative correction to the loop average.

Fig. 2. Contour with a corner.

one angle γ in it. In this case the extra divergent contribution in (4.1) comes from the vicinity of the corner at the point s_0. Expanding $x_\mu(s)$ near s_0 we get

$$f(c) = (\dot{x}_+(s_0)\,\dot{x}_-(s_0)) \int \frac{d\tau\, d\tau'}{(x_+(s_0)\,\tau + x_-(s_0)\,\tau')^2 + a^2} \tag{4.6}$$

plus a contribution from the smooth part of the contour.

The logarithmic contributions coming from these two terms combine to give

$$f(c) = \frac{\pi}{a} L(c) + \{\gamma \cot \gamma - 1\} \log \frac{1}{a} + \text{finite terms} . \tag{4.7}$$

(Here $(\dot{x}_+(s_0)\,\dot{x}_-(s_0)) = \cos\gamma$.) The qualitative nature of the extra term in (4.7) is associated with the violent brensstrahlung that takes place at the corner. This phenomenon plays an important role in the interaction of loops, as we will see later. Let us return now to the discussion of the conjecture (4.5). To perform higher-order calculations it appears to be very helpful to use dimensional regularization of all integrals. By that we mean that the contour itself has a fixed number of dimensions, but the propagator is chosen to belong to $4 - \epsilon$ dimensional space. Divergences, as usual, are replaced by the poles in ϵ. This trick is convenient, since the linear divergence in (4.1) disappears and the conjecture (4.5) is replaced by the simple statement that all poles in ϵ can be absorbed in the charge renormalization. Let us check this in the fourth order of perturbation theory. We have

$$\langle \mathrm{Tr}\, \psi(c) \rangle = 1 + g_0^2 f_1(c) + g_0^4 f_2(c) + \ldots . \tag{4.8}$$

Our conjecture concerning renormalization in this order means that

$$f_2(c) = \frac{b}{\epsilon} f_1(c) + \text{terms finite as } \epsilon \to 0 \tag{4.9}$$

(where $b = \frac{11}{3} C_2(G)\,(16\pi^2)^{-1}$). If (4.9) is true then by redefining the coupling constant we are able to cancel the divergent part of $\langle \mathrm{Tr}\, \psi(c) \rangle$. The computation leading to (4.9) will not be given here. We have verified that (4.9) is true, but will not present the computation here. We merely note that up to fourth order our reformalization conjecture checks. We lack, however, a general proof of it and can present here only some non-rigorous arguments. Let us examine the divergent part of the log $\langle \mathrm{Tr}\, \psi(c) \rangle \equiv W(c)$, which is the effective action for the test particle guided along c. Let us guess that this divergent part depends locally on the trajectory $x(s)$. The linearly divergent part of $W(c)$, $W'_1(c)$, should be parametrization invariant and have dimension one. This will be the case only if

$$W_1(c) = \text{const} \int \sqrt{\dot{x}^2(s)}\, ds = \text{const} \cdot L(c)/a\,. \tag{4.10}$$

As far as the logarithmic divergence is concerned, the only local expression with dimension zero is given by

$$W_0(c) = \text{const}\left(\int \sqrt{\ddot{x}^{\,2}(s)}\, ds\right) \log \frac{1}{a}\,, \tag{4.11}$$

if the parametrization is fixed by the condition $\dot{x}^2(s) = 1$. However, up to now I haven't found any indication for any non-analytic dependence on $\ddot{x}$. So it seems very probable that the logarithmic divergence is absent, however, this argument needs to be considerably improved.

5. Ward identities in the loop space

In this section we derive a set of identities for the Green functions in loop space which follow from the equations of motion (3.8). Since these equations have the form of a conservation law, the corresponding identities for the Green functions will resemble Ward identities in the usual field theory. As before, we begin our discussion by considering two-dimensional chiral fields. In this case, by using the standard trick of shifting variables in the functional integral, it is easy to get the result

$$\begin{aligned} &\frac{\partial}{\partial z_\mu} \langle A^a_\mu(z)\, g(x_1) \times g(x_2) \ldots \times g^+(y_1) \ldots \times g^+(y_n)\rangle \\ &\quad = \sum_k \delta(z - k_k)\langle g(x_1) \ldots \times \lambda^a g(x_k) \ldots g^+(y_n)\rangle \\ &\quad - \sum_k \delta(z - y_k)\langle g(x_1) \ldots g^+(y_k)\, \lambda^a \ldots g^+(y_n)\rangle\,. \end{aligned} \tag{5.1}$$

Here

$$A^a_\mu(z) = \mathrm{Tr}(\lambda^a A_\mu) \equiv \left(\lambda^a g^{-1} \frac{\partial g}{\partial z_\mu}\right),$$

$g(z)$ is a chiral field, λ^a are generators of the group, and $\times$ stands for the direct product. In chiral field theory the Wards identities contain sufficient information to determine the Green functions. That is because in this case, we have the extra condition $g^+(x)\, g(x) = I$. Parametrizing $g(x)$ in some way, and substituting it into (5.1), we get a perturbation expansion for the Green functions. Let us take, for example

$$\begin{aligned} &g(x) = e^{ie_0\phi(x)} = 1 + ie_0\phi + \ldots\,, \\ &A_\mu = \partial_\mu\phi + e_0[\phi, \partial_\mu\phi]\ \ldots \end{aligned} \tag{5.2}$$

where $\phi(x)$ is some hermitian field. Substituting (5.2) into (5.1) we get in the lowest approximation

$$\partial^2 \langle \phi^a(x)\, \phi^b(y) \rangle = \delta_{ab} \delta(x-y)\,, \tag{5.3}$$

indicating that in this approximation we just have free Goldstone bosons. In the next approximation we would obtain the current algebra interaction of these bosons. Higher-order terms are described by Feynman diagrams with massless propagators. In the two-dimensional case we have an infinite number of extra Ward identities associated with the higher currents. On the mass-shell these identities determine the S-matrix almost unambiguously [7]. This should also be the case for the Green functions, but that has not been demonstrated yet.

Our strategy for the gauge fields will be very close to the one described above. We begin with the derivation of Ward identities for the loop averages. The next step will be to derive a gauge-invariant perturbation expansion from these identities. The ultimate purpose of the whole approach is to determine the Green functions in the loop space through the use of the higher Ward identities. This goal has not been achieved yet.

The first important intermediate formula, which we need in order to derive Ward identities, is the following. Let us take some contour C with origin x and let us denote by $\psi(x, y, \mathrm{C})$ the ordered exponent of parallel displacement from the point x to the point y. It is assumed that $y \in \mathrm{C}$. Then, by shifting variables in the functional integral, we obtain

$$\begin{aligned} &\langle \nabla_\mu F^a_{\mu\nu}(z)\, \psi(x, x, \mathrm{C}) \rangle \\ &\quad = \int \mathrm{d}y_\nu \delta(z-y) \langle \psi(x, y, \mathrm{C})\, \lambda^a \psi(y, x, \mathrm{C}) \rangle\,. \end{aligned} \tag{5.4}$$

If we now recall eq. (3.7) we get the desired Ward identity:

$$\begin{aligned} &\frac{\delta}{\delta x_\mu(s')} \langle F^a_\mu(s, \mathrm{c})\, \psi(\mathrm{c}_1) \times \psi(\mathrm{c}_2) \ldots \rangle \\ &\quad + \langle [F_\mu(s', \mathrm{c})\, F_\mu(s, \mathrm{c})]^a \psi(\mathrm{c}_1) \times \psi(\mathrm{c}_2) \ldots\,, \\ &\quad = \delta(s - s') \int \mathrm{d}t \delta(x(s) - y_k(t)) (\dot{x}(s)\, \dot{y}_k(t)) \\ &\quad \cdot \langle \psi(\mathrm{c}_1) \times \ldots \times \lambda^a(s, \mathrm{c})\, \psi(\mathrm{c}_k) \times \ldots \rangle\,. \end{aligned} \tag{5.5}$$

Here

$$\lambda^a(s, \mathrm{c}) = \psi(x, x(s))\, \lambda^a \psi^{-1}(x, x(s))\,.$$

This equation can be further simplified by symmetrizing in s and s'. After that procedure the first term on the right-hand side disappears and we are left with the equation

$$\frac{\delta}{\delta x_\mu(s')} \langle F^a_\mu(s, \mathrm{c})\, \psi(\mathrm{c}_1) \times \ldots \rangle + (s \leftrightarrow s')$$

$$= 2\delta(s - s') \int \mathrm{d}t \delta(x(s) - y_k(t))(\dot{x}(s)\dot{y}_k(t))$$
$$\cdot \langle \psi(c_1) \times ... \times \lambda^a(s, c)\, \psi(c_k) \times ... \rangle \,, \tag{5.6}$$

which is direct analog of (5.1). Let us examine the perturbation expansion that follows from (5.6). To do this, as before we represent

$$\psi(c) = e^{ig_0\phi} = 1 + ig_0\phi - \tfrac{1}{2}g_0^2\phi^2 + ... \,,$$
$$F_\mu(s, c) = \frac{\delta\psi(c)}{\delta x_\mu(s)} \psi^{-1}(c)$$
$$= \frac{\delta\phi}{\delta x_\mu(s)} + ig_0\left[\phi, \frac{\delta\phi}{\delta x_\mu(s)}\right] + ... \,. \tag{5.7}$$

(Here $\phi(c)$ is an hermitian matrix.) In the lowest order we have

$$\frac{\delta^2}{\delta x_\mu(s)\, \delta x_\mu(s')} \langle\phi^a(c)\, \phi^b(\tilde{c})\rangle$$
$$= \delta_{ab}\delta(s - s') \oint_{\tilde{c}} \delta(x(s) - y)\, \dot{x}_\mu(s)\, \mathrm{d}y_\mu \,. \tag{5.8}$$

It is easy to find a solution for (5.8) that plays the role of the bare Green function in the loop space. Namely,

$$\langle\phi^a(c)\phi^b(c)\rangle = \delta_{ab} \oint_c \mathrm{d}x_\mu \oint_{\tilde{c}} \mathrm{d}y_\mu \frac{1}{(x - y)^2} \,. \tag{5.9}$$

There are subtle questions concerning the regularization of the integrals and δ-functions, which we shall examine later. Another equivalent set of equations for the fields in loop space can be obtained from the relations

$$\frac{\delta}{\delta z_\mu(s)} \langle\psi(c)\rangle = \langle F_\mu(s, c)\, \psi(c)\rangle \,,$$
$$\frac{\delta^2}{\delta z_\mu(s)\, \delta z_\mu(s')} \langle\psi(c)\rangle$$
$$= \langle \mathrm{P}(F_\mu(s, c)\, F_\mu(s', c))\, \psi(c)\rangle + \int \mathrm{d}t(\dot{x}(t)\, \dot{x}(s))\, \delta(x(s) - x(t))$$
$$\cdot \langle\lambda^a\psi(c_{1t})\, \lambda^a\psi(c_{2t})\rangle\, \delta(s - s') \,. \tag{5.10}$$

Here the symbol P means the usual ordering in s and s', c_{1t} and c_{2t} are parts of the contour c, which, due to the δ-function in (5.10), necessarily have an intersection point. Eq. (5.10) was obtained by the use of (5.4) and (3.7), Here again we postpone the important discussion of the definition of the δ-function in (5.10). We can simplify eq. (5.10) even more by introducing the operator that picks up the terms containing $\delta(s - s')$ in the second variational derivative. Namely let us define

$$\frac{\partial^2}{\partial x^2(s)} = \lim_{\epsilon\to 0} \int_{-\epsilon}^{+\epsilon} dt \frac{\delta^2}{\delta x_\mu(s + \frac{1}{2}t)\, \delta x_\mu(s - \frac{1}{2}t)} . \tag{5.11}$$

Using (5.11) and (5.10) we get

$$\frac{\partial^2}{\partial x^2(s)} \langle \mathrm{Tr}\ \psi(c)\rangle = \int dt(\dot{x}(s)\, x(t))\, \delta(x(s) - x(t)) \cdot \langle \lambda^a \psi(c_{1t})\, \lambda^a \psi(c_{2t})\rangle . \tag{5.12}$$

A similar equation was independently obtained by Makeenko and Migdal [9] using a somewhat different approach, who also found the large-N limit for it. It is still a question whether the operator $\partial^2/\partial x^2(s)$ contains sufficient information about the second variational derivative.

Now we turn to the analysis of the δ-function in (5.12). The definition of this δ-function depends on the regularization scheme. If we admit dimensional regularization, then $\langle \mathrm{Tr}\ \psi(c)\rangle$ is presumably finite [after charge renormalization (see sect. 4)] and to get finite results on the right-hand side of eq. (5.12) we have to define

$$\delta(x(s) - x(s')) = \lim_{\Delta\to d} (d - \Delta) \frac{1}{|x(s) - x(s')|^\Delta} , \tag{5.13}$$

where d is the dimensionality of space and the expression on the right-hand side should be understood in the sense of analytic continuation in Δ (see ref. [10]). Let us first examine the case $d = 2$. In order to understand the meaning of (5.13) we have to integrate it with smooth test functions $\phi(s)$ and $\chi(s)$. We have

$$\psi(s)\, \chi(x')\, \delta(x(s) - x(s'))\, ds\, ds' = \lim_{\Delta\to 2} (2 - \Delta) \int \frac{\psi(s)\, \chi(s')\, ds\, ds'}{|x(s) - x(s')|^\Delta} . \tag{5.14}$$

Singularities in Δ in (5.14) arise from the points where $\chi(s) = \chi(s')$. We now have to distinguish two cases. Let us first assume that the contour is simple, i.e., it has no self-intersections and it is smooth everywhere. We also use the parametrization (4.3). Changing variables we get

$$\int \psi(s)\, \chi(s')\, \delta(x(s) - x(s'))\, ds\, ds' = \lim_{\Delta\to 2} (2 - \Delta) \int \frac{\psi(s)\, \chi(s + t)\, ds\, dt}{|t|^\Delta} \cdot (1 + O(t^2)) . \tag{5.15}$$

The integral in (5.15) does not have a singularity at $\Delta = 2$ if it is understood as an analytical continuation. Hence, for simple contours $\delta(x(s) - x(s')) = 0$ (no self-intersections). If there is a self-intersection, say, if $x(s_1) = x(s_2)$, the result is different. A non-zero contribution comes from the neighborhood of the self-intersection point. Taking $s = s_1 + \tau$ and $s' = s_2 + \tau'$ and expanding in τ and τ' we obtain

$$\int \psi(s)\, \chi(s')\, \delta(x(s'))\, \mathrm{d}s\, \mathrm{d}s'$$

$$\approx \psi(s_1)\, \chi(s_2) \int \frac{\mathrm{d}\tau\, \mathrm{d}\tau'}{(\dot{x}(s_1)\tau + \dot{x}(s_2)\tau')^{\Delta}} (2 - \Delta) + (s_1 \leftrightarrow s_2)\,. \tag{5.16}$$

An easy evaluation of the integral gives

$$\delta(x(s) - x(s')) = |\epsilon_{\mu\nu} \dot{x}_\mu(s_1)\, \dot{x}_\nu(s_2)|^{-1}$$
$$\times \{\delta(s - s_1)\, \delta(s' - s_2) + \delta(s' - s_2)\, \delta(s - s_1)\}$$
$$\equiv \frac{1}{|\sin \gamma|} (\delta(s - s_1)\, \delta(s' - s_2) + \delta(s' - s_2)\, \delta(s - s_1)) \tag{5.17}$$

(here γ is the angle of the self-intersection).

In the four-dimensional case ($d = 4$) similar manipulations give the result

$$\delta(x(s) - x(s')) = 0\,, \quad \text{(without self-intersections)}\,, \tag{5.18}$$

$$\delta(x(s) - x(s')) = a_1 \{\ddot{\delta}(s - s_1)\, \delta(s' - s_2) + \delta(s - s_1)\, \ddot{\delta}(s' - s_2)\}$$
$$+ a_2 \dot{\delta}(s - s_1)\, \dot{\delta}(s' - s_2) + (s \leftrightarrow s')\}\,,$$
$$\text{(with one self-intersection)}\,.$$

So, we see that the contour δ-function is a well-defined object (with self-interactions). However, in the derivation of (5.12) we conjectured implicitly that the first term in (5.10) does not contain $\delta(s - s')$. That is certainly true if we deal with the cutoff version of the theory, but it is far from obvious for the dimensionally regularized version. To clarify the question one has to analyze the short-distance expansion of this term and to use the definition of the δ-function described above. We have not completed this analysis in the four-dimensional case, but it seems possible that for the case of simple contours this term does not contribute. If it does not, we get

$$\frac{\partial^2}{\partial x^2(s)} \langle \mathrm{Tr}\, \psi(\mathrm{c}) \rangle = 0 \tag{5.19}$$

(for the simple contours). Let us stress again, that (5.19) may be untrue since it is based on an unproved conjecture concerning short-distance behavior. It is of some interest, nevertheless, to point out that there exists a rather general solution of it. Namely, it is easy to check that

$$\frac{\partial^2}{\partial x^2(s)} (f(\sigma_{\mu\nu}^2)) = 0\,, \tag{5.20}$$

where

$$\sigma_{\mu\nu} = \int_c x_\mu \, \mathrm{d}x_\nu .$$

For the contours with self-intersections this ansatz has to be modified.

It is clear that a much more detailed investigation of eq. (5.10) is necessary before we can effectively exploit it.

6. Unsolved problems and possible ways to solve them

This paper should be considered only as a proposal for a future theory. It is appropriate therefore to discuss here unsolved problems and perspectives of future investigations.

The most important question that remains unanswered is whether a hidden symmetry is present in the four-dimensional gauge theories. We have to build up the Lax representation analogous to (3.14). Though it has not been completed, I would like to introduce some mathematical notions, which should be useful for this task. This is the concept of the "area differentiation", $\delta/\delta\sigma_{\mu\nu}(s)$. Geometrically this operation is obvious, and, as a matter of fact, has already been used by Mandelstam [11]. It is defined as a change of the functional if we attach to the contour an infinitesimal area $\delta\sigma_{\mu\nu}$. However, we need an analytic definition of this derivative which, to my knowledge, has never appeared in literature. The definition is as follows. Let us consider expansion of the second variational derivative of some functional $\phi(\mathrm{c})$

$$\frac{\delta^2\phi(\mathrm{c})}{\delta z_\mu(s)\,\delta z_\nu(s')} = \tfrac{1}{2}(N_{\mu\nu}(s,\mathrm{c}) + N_{\mu\nu}(s',\mathrm{c}))\,\dot{\delta}(s-s') + M_{\mu\nu}(s,\mathrm{c})\,\delta(s-s') + \text{regular terms} \tag{6.1}$$

(here $\dot{\delta}$ is derivative of the δ). We define

$$\frac{\delta\phi(\mathrm{c})}{\delta\sigma_{\mu\nu}(s)} \stackrel{\mathrm{def}}{=} N_{\mu\nu}(s,\mathrm{c}) . \tag{6.2}$$

It is clear from (6.2) that $N_{\mu\nu}(s,\mathrm{c})$ is an antisymmetric tensor. Another property, which is easy to check, is

$$\frac{\delta}{\delta\sigma_{\mu\nu}(s)}(\phi_1(\mathrm{c})\,\phi_2(\mathrm{c})) = \phi_1\frac{\delta\phi_2}{\delta\phi_{\mu\nu}} + \frac{\delta\phi_1}{\delta\sigma_{\mu\nu}}\phi_2 \tag{6.3}$$

and, using the condition

$$\dot{z}_\mu(s)\,\frac{\delta\phi(\mathrm{c})}{\delta z_\mu(s)} = 0 , \tag{6.4}$$

we get

$$\dot{z}_\mu(s)\, N_{\mu\nu}(s, c) \overset{\text{def}}{=} z_\mu \frac{\delta\phi(c)}{\delta\sigma_{\mu\nu}(s)} = \frac{\delta\phi(c)}{\delta z_\nu(s)}\,, \tag{6.5}$$

as should be expected from the geometrical picture. An equivalent definition (*cf.* eq. (5.11)) is

$$\frac{\delta}{\delta\sigma_{\mu\nu}(s)} = \lim_{\epsilon\to 0} \int_{-\epsilon}^{+\epsilon} \mathrm{d}\tau\tau \frac{\delta^2}{\delta z_\mu(s+\frac{1}{2}\tau)\,\delta z_\nu(s-\frac{1}{2}\tau)}\,. \tag{6.6}$$

If relation (6.4) does not hold, then (6.6) must be modified. It seems probable that one should look for the Lax representation in the form

$$\left(\frac{\delta}{\delta\sigma_{\mu\nu}(s)} + \gamma\epsilon_{\mu\nu\lambda\gamma}\frac{\delta}{\delta\sigma_{\lambda\gamma}(s)}\right)\phi = F_{\mu\nu}\phi\,, \qquad \phi = \phi(s, \mathrm{C}, \gamma)\,. \tag{6.7}$$

Relation (6.7) is only a tentative form, which is not completely defined since ϕ does not satisfy (6.4), and we do not know at the moment how to define an area differentiation applicable to $\phi(s, \mathrm{C}, \gamma)$. Furthermore it is not clear that (6.7) will be really equivalent to the equations of motion. It remains to be demonstrated that (6.7) or some similar relation really makes sense. The next interesting mathematical problem in the three-dimensional case is to construct explicitly the functional $\chi(s, c)$ in (3.10) and also to investigate higher conserved currents and Bäcklund transformations in loop space. We have no definite proposals on this topic at present.

It seems very important to develop further the manifestly gauge-invariant perturbation theory outlined in sect. 5. There should exist a convenient and transparent diagram technique describing propagation of the bare closed strings. In contrast with usual diagrams we shall have summation over all possible contours instead of the usual integrations.

Next, let us discuss the question of instantons in the string context. In the two-dimensional σ-models it is known [12] that point-like instantons disturb the system in such a way as to create a finite correlation length. In four-dimensional gauge theories it is natural to expect that the major disordering effect will come from string-like instantons. Such instantons could be envisaged as a combination of usual instantons, ordered along some curve. They also can be considered as a generalized monopole solution (the usual monopole solution is represented by a straight line in four-dimensional space). It has been shown that in the abelian theory such string-like instantons indeed lead to confinement for large enough coupling [13]. In the non-abelian case it is an open question. Further investigation of string-like instantons seems very important to me.

Just as in the usual field theories the most important role is played by the free particle propagator, in gauge theories we have to consider the propagator for the free strings. An attempt was made to introduce such propagators in dual string theories, but they were correctly defined only for $D = 26$. I want to point out the direction one might take to resolve this problem. Symbolically, the propagator is represented by

$$G(\mathrm{c}) = \sum_{(\mathrm{s})} \mathrm{e}^{-A(\mathrm{s})} , \tag{6.8}$$

where the summation goes over all possible surfaces, which have the contour c as a boundary, and $A(\mathrm{s})$ is the area of the surface s. The surfaces are supposed to be free, i.e., not to have self-intersections, double covering, etc. The question now is how to transform (6.8) into some real formula. For the usual particles we know the answer: namely, the standard expression for the particle propagator is

$$G(\boldsymbol{x} - \boldsymbol{x}') = \sum_{(\mathrm{P})} \mathrm{e}^{-L(\mathrm{P})} , \tag{6.9}$$

where one sums over all free paths connecting $\boldsymbol{x}$ to $\boldsymbol{x}'$. This can be rewritten as

$$G(\boldsymbol{x} - \boldsymbol{x}') = \int_0^\infty \mathrm{d}\tau \, \mathrm{e}^{-m^2\tau} \int_{\substack{x(0)=x \\ x(\tau)=x'}} \mathcal{D}x(\tau) \exp[-\int_0^\tau \tfrac{1}{2}\dot{x}^2 \, \mathrm{d}\tau] . \tag{6.10}$$

The proper time representation used in (6.10) is very convenient when using Feynman diagrams. We would now like to find an analogous expression for $G(\mathrm{c})$. Let us introduce the complex plane z, and consider some region M_Γ on it, with boundary Γ. Let us next introduce the functional integral

$$F(z(s), \boldsymbol{x}(s)) = \int \mathcal{D}x(z) \exp-\left\{ \int_{\mathrm{M}_\Gamma} \left| \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}z} \right|^2 \mathrm{d}^2 z \right\} ,$$

$$\boldsymbol{x}(z(s)) = \boldsymbol{x}(s) \tag{6.11}$$

Here we parametrize Γ by $z = z(s)$, and c by $\boldsymbol{x} = \boldsymbol{x}(s)$. The integral can be expressed through the Dirichlet-Green function of M_Γ. The important point is that F satisfies the equations

$$\dot{z}_a(s) \frac{\delta F}{\delta z_a(s)} + \dot{x}_\mu(s) \frac{\delta F}{\delta x_\mu(s)} = 0 ,$$

$$\epsilon_{ab} \dot{z}_a \frac{\delta F}{\delta z_b} = \frac{\delta^2 F}{\delta x_\mu^2(s)} - \dot{x}^2(s) F , \tag{6.12}$$

which are consequences of

$$\dot{z}_a \frac{\delta W}{\delta z_a} + \dot{x}_\mu \frac{\delta W}{\delta x_\mu} = 0 ,$$

$$\epsilon_{ab} \dot{z}_a \frac{\delta W}{\delta z_b} = \left(\frac{\delta W}{\delta x_\mu} \right)^2 - \dot{x}_\mu^2 , \tag{6.13}$$

where

$$W = \int_{M_\Gamma} \left| \frac{dx}{dz} \right|^2 d^2z \, .$$

We see now the complete analogy between particles and strings. The role of proper time is played by the contour Γ, meaning that each point of the string has its own proper time. To get the propagator we have to integrate F over all possible Γ. It is not clear at the moment how to define this summation in order to get the correct boundary conditions. However, since we treat longitudinal degrees of freedom properly I don't think that the problem with $D = 26$ arises. It remains to be proved, however.

Let us now try to formulate what are our expectations and hopes concerning the future theory. First one has to find explicitly higher conservation laws in the four-dimensional case. Then, using the Ward identities associated with these conservation laws, one has to derive constraints on the Green functions in loop space. Presumably, these constraints will be strong enough to determine completely these Green functions or S-matrix elements associated with them. The latter indeed happens in the two-dimensional chiral theories. If all this fantasy becomes real, gluon dynamics will be solved. The question of matter and its coupling to gluons will still remain, but it seems to me that although we are on the right track with the interactions of gluons, we don't have any understanding of the matter multiplets or of their couplings. It might therefore be premature to speculate on their dynamics. Finally, I would like to point out that there are many valuable papers which are relevant for the problems discussed above and they might be useful for the future investigation of these problems. An incomplete list of such papers, together with those already quoted, is given in refs. [14–22].

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References

[1] A.M. Polyakov, Phys. Lett. 82B (1979) 247.
[2] K. Wilson, Phys. Rev. D10 (1974) 2445.
[3] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.
[4] K. Pohlmeyer, Comm. Math. Phys. 46 (1976) 207.
[5] V. Zakharov and A. Mikhailov, Zh ETF Pisma 27 (1978) 47.
[6] A. Zamolodchikov and Al. Zamolodchikov, Nucl. Phys. B133 (1978) 525.
[7] M. Lüscher, Nucl. Phys. B135 (1978) 1.
[8] A. Polyakov, Phys. Lett. 72B (1977) 224.
[9] Yu. Makeenko and A.A. Migdal, Phys. Lett. B, to be published.
[10] I. Gelfand and G. Shilov, Generalized functions (Academic Press, New York, 1964).
[11] S. Mandelstam, Phys. Rev. 175 (1968) 1580.

[12] I. Frolov, V. Fateev and A. Schwartz, ZhETF Pisma 28 (1978) 5, 274.
[13] A.M. Polyakov, Phys. Lett. 59B (1975) 80.
[14] Y. Nambu, Phys. Lett. 80B (1979) 372.
[15] J.L. Gervais and A. Neveu, Phys. Lett. 80B (1979) 255.
[16] E. Corrigan and B. Hasslacher, Phys. Lett. 81B (1979) 181.
[17] C. Marshall and P. Ramond, Nucl. Phys. B85 (1975) 375.
[18] C.N. Yang, Phys. Rev. Lett. 33 (1974) 445.
[19] R. Miulos and Ya. Sinai, Math. Sbot. 18 (1967).
[20] F. Gliozzi, T. Regge and M. Virasoro, Phys. Lett. 81B (1979) 178.
[21] M. Virasoro, Phys. Lett. 82B (1979) 436.
[22] A.A. Migdal, to be published.